Econ 802

Answers to Midterm 1

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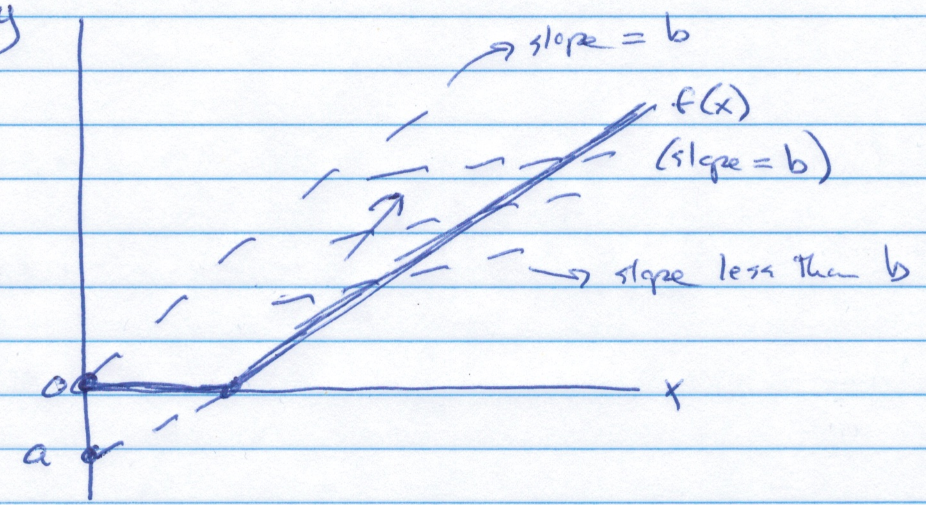
1(a) When $a<0$, we have an interval of inputs where $y=0$. Suppose $a+bx>0$ so $y>0$. We have increasing returns if $f(tx) > tf(x)$ for $t>1$. This requires

$$(a+btx) > t[a+bx] \Rightarrow a > ta \quad \text{or} \quad 1<t$$

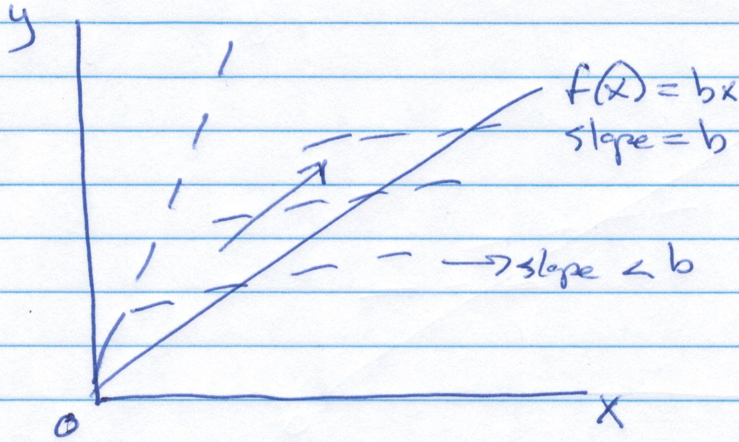
because $a<0$. Thus we have increasing returns to scale.



An isoprofit line has $\pi = py - wx$ for some fixed π. $\Rightarrow y = \frac{\pi + wx}{p} \Rightarrow$ slope $= \frac{w}{p}$. If $\frac{w}{p} < b$ then there is no solution because we can always move to a higher isoprofit line (see flat dashed lines). If $\frac{w}{p} = b$ there is a unique solution at $(0,0)$ (see steeper dashed line). The same is true if $\frac{w}{p} > b$.

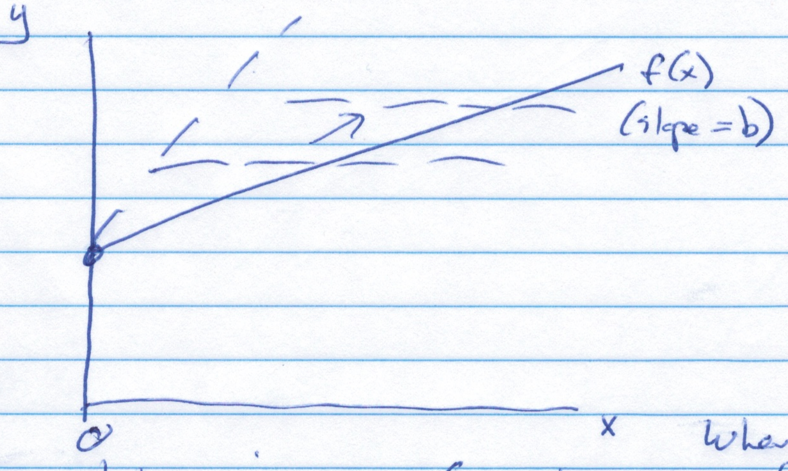
(b) When $a=0$ we have $y = \max\{0, bx\} = bx$ because $x \geq 0$ and $b>0$. Thus $f(tx) = tf(x)$ for all $t>0$ due to $b(tx) = t(bx)$. So we have constant returns to scale.

1(b) cont.



As in part (a) if $\frac{w}{p} < b$ there is no solution because we can always get to a higher isoprofit line. If $\frac{w}{p} = b$ the slopes of the isoprofit lines are parallel to the production function and the highest isoprofit line is the one that coincides with this function. So every point on the production function is a solution (and they all give zero profit). If $\frac{w}{p} > b$ then the isoprofit lines are steeper than b and the highest profit occurs uniquely at the origin (see the steep dashed line through the origin).

1(c) When $a > 0$ we have $y = a + bx$ for all $x \geq 0$
Decreasing returns occurs if $f(tx) < tf(x)$ for $t > 1$ or $a + b(tx) < t(a+bx) \Rightarrow a < ta$ which is true for $a > 0$.
So we have DRS.



As before there is no solution if $\frac{w}{p} < b$. When $\frac{w}{p} = b$ the highest isoprofit line coincides with the production function and every $x \geq 0$ is profit max.

Note: Maybe there is a fixed input that gives $y > 0$ even when $x = 0$.

When $\frac{w}{p} > b$ the unique solution is $x = 0$ (see steep isoprofit line). However, this gives positive output and positive profit.

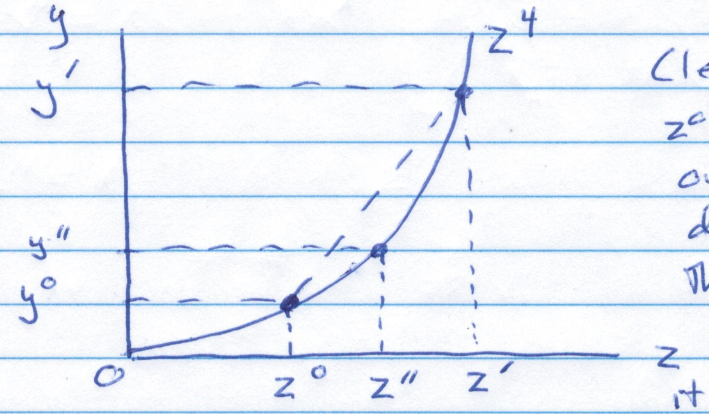
2(a) Consider two input vectors $x^0 \neq x^1$. These give the intermediate inputs $z^0 = a x_1^0 + b x_2^0$ and $z' = a x_1' + b x_2'$

Now consider a convex combination $x'' = t x^0 + (1-t) x'$ (where $0 < t < 1$) which gives

$$z'' = a x_1'' + b x_2'' = a\left[t x_1^0 + (1-t) x_1'\right] + b\left[t x_2^0 + (1-t) x_2'\right]$$

$$\Rightarrow \quad z'' = t z^0 + (1-t) z'.$$

Finally consider the relationship $y = z^4$:



Clearly z'' is between z^0 and z', and yields an output y'' below the dashed line. This shows that Y is not convex. If Y were convex, it would be true that any convex combination of $(y^0 - x^0)$ and $(y' - x')$ would also be in Y. However, ~~[illegible]~~ a convex combination of $-x^0$ and $-x'$ actually gives an output y'' less than the convex combination of y^0 and y'. Since Y is not convex, it cannot be strictly convex either.

(b) FOC:

$$\left.\begin{aligned} 4pa(ax_1 + bx_2)^3 - w_1 &= 0 \\ 4pb(ax_1 + bx_2)^3 - w_2 &= 0 \end{aligned}\right\} \quad 4p(ax_1 + bx_2)^3 = \frac{w_1}{a} = \frac{w_2}{b}$$

In general it will not be possible to satisfy FOC because the parameters will not typically satisfy $\frac{w_1}{a} = \frac{w_2}{b}$.
Interpretation: There is a linearity built into the production of the z good that usually leads to boundary solutions (like with cost min when we have linear technology).

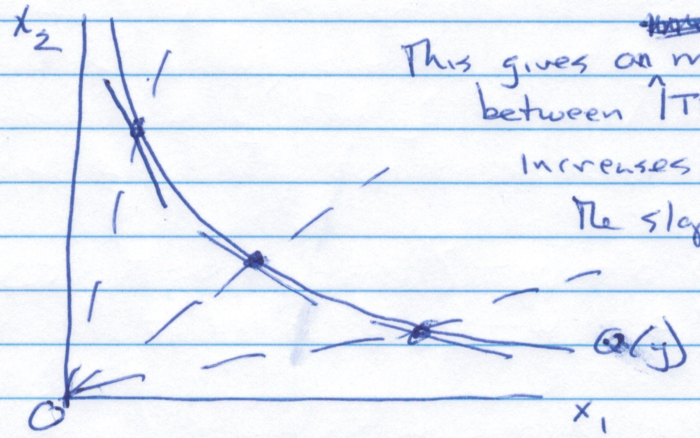
2(c) Continued SOC (necessary). The second derivative with respect to x_1 is $12pa^2(ax_1+bx_2)^2$. The second derivative with respect to x_2 is $12pb^2(ax_1+bx_2)^2$. The cross partial is $12pab(ax_1+bx_2)^2$. So the Hessian is

$$\underbrace{12p(ax_1+bx_2)^2}_{+} \cdot \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix} \quad \text{where } a>0,\ b>0$$

But the nec. SOC requires this to be negative semi-definite which implies non-positive diagonal elements, which is <u>false</u> (unless $x_1 = x_2 = 0$, when we get a matrix of zeroes).

Interpretation: The second stage of the production process is $y = z^4$, which has increasing returns to scale. This is not compatible with the necessary SOC.

3(a) Suppose we have strictly convex input requirement sets (or a strictly quasi-concave production function). Then the isoquant curves this way:



This gives a monotonic relationship between $|TRS|$ and $\frac{x_2}{x_1}$; as $|TRS|$ increases along the isoquant $Q(y)$, the slope $\frac{x_2}{x_1}$ increases so the slope $\frac{x_1}{x_2}$ decreases.

Assuming this relationship is differentiable it would make sense to write $\frac{\partial\left(\frac{x_1}{x_2}\right)}{\partial |TRS|} < 0$ for movements along a given isoquant.

3(b) We know the slope of the isoquant through a point x is $-\frac{f_1(x)}{f_2(x)}$ where $f(x)$ is the production function and $f_1(x)$, $f_2(x)$ are marginal products.

Let's carry out the calculation:

$$f_1(x) = \left(\tfrac{1}{a}\right)(x_1^a + x_2^a)^{\frac{1}{a}-1} a x_1^{a-1}$$

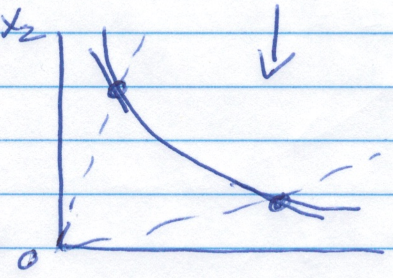
$$f_2(x) = \left(\tfrac{1}{a}\right)(x_1^a + x_2^a)^{\frac{1}{a}-1} a x_2^{a-1}$$

$$\Rightarrow -\frac{f_1(x)}{f_2(x)} = -\left(\frac{x_1}{x_2}\right)^{a-1} \quad \text{so } TRS = -\left(\frac{x_2}{x_1}\right)^{1-a}$$

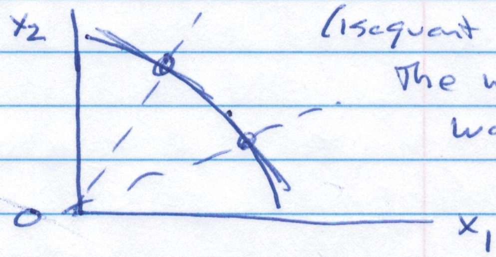
$$\text{or } |TRS| = \left(\frac{x_2}{x_1}\right)^{1-a}$$

The parameter a determines how rapidly TRS changes in response to the input ratio $\frac{x_2}{x_1}$.

When $a<1$, $\frac{x_2}{x_1}\uparrow$ implies $|TRS|\uparrow$ so we get this type of curvature:



When $a>1$, $\frac{x_2}{x_1}\uparrow$ implies $|TRS|\downarrow$ and we get this:



(isoquant bends the wrong way)

For $a \to +1$ we approach linear isoquants.

For $a \to -\infty$ we approach Leontief isoquants.

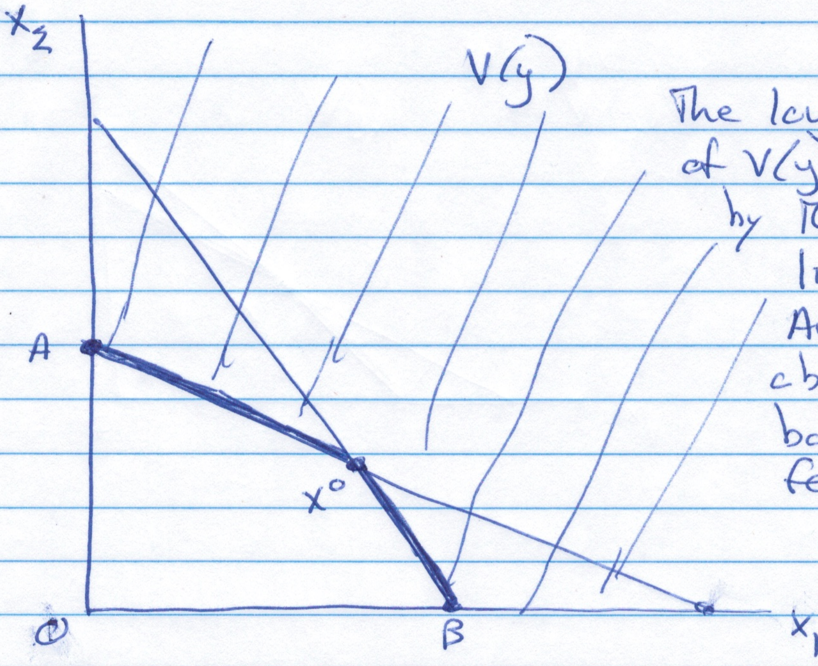
(c) From part (b) we will have $\frac{w_1}{w_2} = \frac{f_1(x)}{f_2(x)} = \left(\frac{x_2}{x_1}\right)^{1-a}$ | from FOC in the CES case.

Now consider the Cobb-Douglas function $y = f(x) = x_1^{\alpha} x_2^{\beta}$ with $f_1(x) = \alpha x_1^{\alpha-1} x_2^{\beta}$ and $f_2(x) = \beta x_1^{\alpha} x_2^{\beta-1}$.

from FOC | This gives $\frac{w_1}{w_2} = \frac{f_1(x)}{f_2(x)} = \left(\frac{\alpha}{\beta}\right)\left(\frac{x_2}{x_1}\right)$ or $\frac{x_2}{x_1} = \left(\frac{\beta}{\alpha}\right)\left(\frac{w_1}{w_2}\right)$

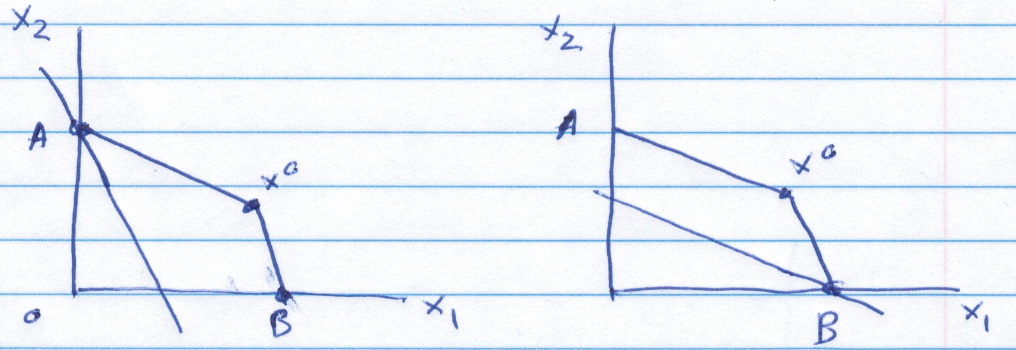
If we have $a = 0$ (elasticity of substitution $= 1$) and $\alpha = \beta$, then we get $\frac{x_2}{x_1} = \frac{w_1}{w_2}$ in both cases. This occurs because the Cobb-Douglas is a special case of the CES.

4(a)



The lower boundary of $V(y)$ is indicated by the heavy line segments. Any point on or above the lower boundary is a feasible way to produce y because it is always possible to use one of the two production functions to get some $y' \geq y$ and then any extra output can be thrown away if necessary. $V(y)$ is <u>monotonic</u> because if $x \in V(y)$ and $x' \geq x$ then $x' \in V(y)$ (all points to the northeast of x are also in the set). It is <u>not</u> convex (for example, a convex combination of A and B is not in $V(y)$). It is <u>closed</u> as long as the relevant parts of the axes are included.

(b)



If the isocost lines are steeper than the slope between A and B the firm chooses A. If they are flatter than this slope the firm chooses B. If they have the same slope the firm is indifferent between A and B (but does not choose any other point)

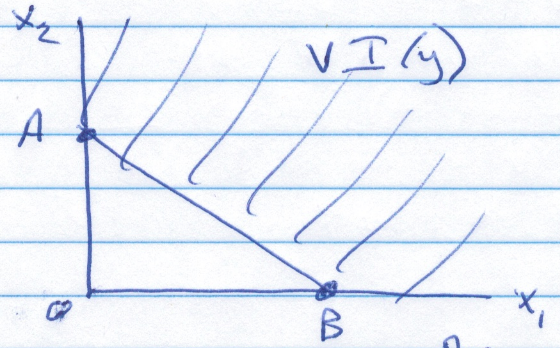
To express this more systematically: The slope of an isocost line is $-\frac{w_1}{w_2}$. Let $\left(\frac{w_1}{w_2}\right)^*$ be the price ratio where the isocost lines have slope equal to the slope between A and B (that is, $\left(\frac{w_1}{w_2}\right)^* = \frac{x_2^A}{x_1^B}$).

When $\frac{w_1}{w_2} > \left(\frac{w_1}{w_2}\right)^*$, $x_1(w,y) = 0$, $x_2(w,y) = x_2^A$

When $\frac{w_1}{w_2} = \left(\frac{w_1}{w_2}\right)^*$ either we have the solution above or the solution below (non-uniqueness)

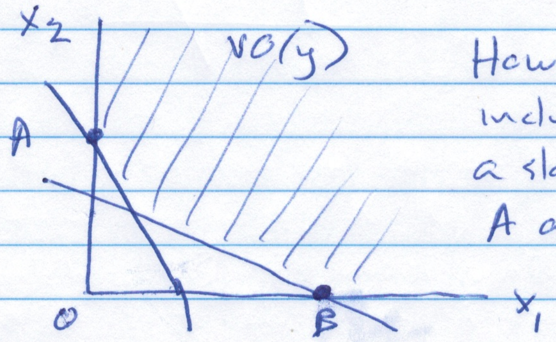
When $\frac{w_1}{w_2} < \left(\frac{w_1}{w_2}\right)^*$, $x_1(w,y) = x_1^B$, $x_2(w,y) = 0$

(c) Every price vector will give either point A or point B so these are the only quantity vectors that will ever be observed. We usually require VI(y) to be convex, monotonic, and closed, so Professor Z is likely to have



Note: The professor will not notice the non-convexity of the true V(y) set because the firm never operates at an interior point.

VO(y) will include the points on or above all of the isocost lines that have been observed. For two observations it might look like this:



However, with many observations including some isocost lines with a slope close to the slope between A and B, VO(y) will be close to VI(y).

5(a) At the prices p^*, the firm uses y^* and has profit p^*y^*. If the firm is maximizing profit, then

$$p^*y^* \geq p^*y \quad \text{for all } y \in Y$$

When the prices are tp^*, the firm uses y'. If this is profit maximizing, then

$$(tp^*)y' \geq (tp^*)y \quad \text{for all } y \in Y$$

where $t > 0$. This implies

$$p^*y' \geq p^*y \quad \text{for all } y \in Y.$$

These inequalities can all be true if and only if

$$p^*y^* = p^*y' \quad \text{where } y^* \neq y'.$$

This is possible, it just means that at the prices p^*, the optimal production plan is non-unique.

(b) Because all inputs have positive prices a cost-minimizing firm will never use more of any input than the minimum required to achieve y. Therefore

$$y = a_1x_1 = a_2x_2 = \ldots = a_nx_n$$

So $\quad x_i = \dfrac{y}{a_i} \quad$ for all $i = 1..n$

Thus the conditional input demands are

$$x_i(w, y) = \frac{y}{a_i} \quad \text{for all } i = 1..n$$

(note that the input demands do not depend on the price vector w).

The cost function is $c(w,y) = \sum_{i=1}^{n} w_i x_i(w,y)$

$$\Rightarrow \quad c(w,y) = y \sum_{i=1}^{n} \frac{w_i}{a_i}$$

5(c) (i) FOC

Advantages: if you make enough assumptions you can show that the substitution matrix is negative definite. This implies that the unconditional input demand curves are downward sloping.

Disadvantages: you need differentiability of the production function, and also negative definiteness of the Hessian matrix in order to use the implicit function theorem and invert a matrix. This method is also computationally burdensome.

(ii) Algebraic

Advantages: This method requires the fewest assumptions (only Weak Axiom of Profit Max). You don't need differentiability, monotonicity, convexity, etc. Also it applies globally, not just at a local maximum

Disadvantages: The results are relatively weak. For example you can show that input demand curves are non-increasing but not that they are decreasing. You also don't get symmetry, negative definiteness, etc

(iii) Hotelling

Advantages: don't need neg. definiteness of the Hessian and don't need to invert a matrix.

Disadvantages: you need the profit function to be differentiable. Also you can only prove that the substitution matrix is negative semi-definite, not that it is neg. definite.